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On Lorentz Invariance, Spin-Charge Separation And SU(2) Yang-Mills Theory

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Previously it has been shown that in spin-charge separated SU(2) Yang-Mills theory Lorentz invariance can become broken by a one-cocycle that appears in the Lorentz boosts. Here we study in detail the structure of this one-cocycle. In particular we show that its non-triviality relates to the presence of a (Dirac) magnetic monopole bundle. We also explicitely present the finite version of the cocycle.

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Recently the properties of four dimensional SU(2) Yang-Mills theory have been investigated using spin-charge separated variables [1], [2] that might describe the confining strong coupling regime of the theory [3]. For example, it was shown that even though these variables reveal the presence of two massless Goldstone modes, this apparent contradiction with the existence of a mass gap becomes resolved since these Goldstone modes break Lorentz invariance by a one-cocycle [1]: The ground state must be Lorentz invariant, thus the one-cocycle is to be removed. This demand fixes the ground state uniquely and deletes all massless states from the spectrum [1].

In [1] only the infinitesimal form of the one-cocycle was presented. Here we display its finite form. We also verify that the one-cocycle is indeed non-trivial, by relating it to the nontriviality of the Dirac magnetic monopole bundle. For definiteness we develop our arguments in the four dimensional space $\mathbb{R}^4$ with Euclidean signature. The extension from $SO(4)$ to $SO(3,1)$ is straightforward.

Locally, in the Maximal Abelian Gauge the spin-charge separation amounts to the following decomposition of the off-diagonal components $A^{\pm}_\mu$ of the gauge field A^a_μ [1], [3].

$$A^+_\mu \;=\; A^1_\mu + iA^2_\mu \;=\; \psi_1 e_\mu + \psi_2 e^\star_\mu \tag{1}$$

where the spin field e_μ

$$e_\mu = \frac{1}{\sqrt{2}}(e^1_\mu + ie^2_\mu)$$

is normalized according to

$$\begin{aligned} e_\mu e_\mu &= 0 \\ e_\mu e^\star_\mu &= 1 \end{aligned} \tag{2}$$

This can be viewed as a Clebsch-Gordan type decomposition of $A^{\pm}_\mu$, when interpreted as a tensor product of the complex spin-variable e_μ that remain intact under SU(2) gauge transformations and the charge variables $\psi_{1,2}$ that are Lorentz scalars but transform under SU(2); see [1] for details.

The decomposition introduces an internal $U_I(1)\times\mathbb{Z}_2$ symmetry that is not visible to A^a_μ. The $U_I(1)$ action is

$$U_I(1): \qquad \begin{array}{rcl} e_\mu & \to & e^{-i\lambda} e_\mu \\ \psi_1 & \to & e^{i\lambda}\psi_1 \\ \psi_2 & \to & e^{-i\lambda}\psi_2 \end{array} \tag{3}$$

This is a local frame rotation, in particular it preserves the orientation in e_μ. The $\mathbb{Z}_2$ action exchanges ψ_1 and ψ_2,

$$\mathbb{Z}_2: \qquad \begin{array}{rcl} e_\mu & \to & e^\star_\mu \\ \psi_1 & \to & \psi_2 \\ \psi_2 & \to & \psi_1 \end{array} \tag{4}$$

This changes the orientation on the two-plane spanned by e_μ. (The realization of $\mathbb{Z}_2$ is unique only up to phase factor.)

In the Yang-Mills action the complex scalar fields $\psi_{1,2}$ becomes combined into the three component unit vector [1]

$$\mathbf{t} \;=\; \frac{1}{\rho^2}\begin{pmatrix}\psi_1^\star & \psi_2^\star\end{pmatrix}\vec{\sigma}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \;=\; \frac{1}{\rho^2}\begin{pmatrix}\psi_1^\star\psi_2+\psi_2^\star\psi_1\\ i(\psi_1\psi_2^\star-\psi_2\psi_1^\star)\\ \psi_1^\star\psi_1-\psi_2^\star\psi_2\end{pmatrix} \;=\; \begin{pmatrix}\cos\phi\sin\theta\\ \sin\phi\sin\theta\\ \cos\theta\end{pmatrix} \tag{5}$$

We have here parameterized

$$\begin{aligned}\psi_1 &= \rho\, e^{i\zeta}\cos\tfrac{\theta}{2}\, e^{-i\phi/2}\\ \psi_2 &= \rho\, e^{i\zeta}\sin\tfrac{\theta}{2}\, e^{i\phi/2}\end{aligned} \tag{6}$$

The internal $U_I(1)$ transformation sends

$$t_\pm \;=\; \frac{1}{2}(t_1 \pm i t_2) \;\to\; e^{\mp 2i\lambda} t_\pm \tag{7}$$

but t_3 remains intact. The $\mathbb{Z}_2$ action is a rotation that sends $(t_1,t_2,t_3)\to(t_1,-t_2,-t_3)$. In terms of the angular variables in (5) this corresponds to $(\phi,\theta)\to(2\pi-\phi,\pi-\theta)$. Thus we may opt to eliminate the $\mathbb{Z}_2$ degeneracy by a restriction to the upper hemisphere $\theta\in[0,\frac{\pi}{2})$.

The off-diagonal components (1) determine the embedding of a two dimensional plane in $\mathbb{R}^4$. The space of two dimensional linear subspaces of $\mathbb{R}^4$ is the real Grassmannian manifold $Gr(4,2)$ [4], it can be described by the anti-symmetric tensor [1], [5], [6]

$$P_{\mu\nu} = \frac{i}{2}(A_\mu^+ A_\nu^- - A_\nu^+ A_\mu^-) \;=\; A_\mu^1 A_\nu^2 - A_\nu^1 A_\mu^2 \tag{8}$$

that obeys the Plücker equation

$$P_{12}P_{34} - P_{13}P_{24} + P_{23}P_{14} = 0 \tag{9}$$

Conversely, *any* real antisymmetric matrix $P_{\mu\nu}$ that satisfies (9) can be represented in the functional form (8) in terms of some two vectors A_μ^1 and A_μ^2. The Plücker equation describes the embedding of $Gr(4,2)$ in the five dimensional projective space $\mathbb{RP}^5$ as a degree four hypersurface [4], a homogeneous space

$$Gr(4,2) \simeq \frac{SO(4)}{SO(2)\times SO(2)} \;\simeq\; \mathbb{S}^2\times\mathbb{S}^2 \tag{10}$$

When we substitute (1) we get

$$P_{\mu\nu} = \frac{i}{2}(|\psi_1|^2 - |\psi_2|^2)\cdot(e_\mu e_\nu^\star - e_\nu e_\mu^\star) \;=\; \frac{i}{2}\cdot\rho^2\cdot t_3\cdot(e_\mu e_\nu^\star - e_\nu e_\mu^\star) \;=\; \rho^2\cdot t_3 H_{\mu\nu} \tag{11}$$

This is clearly invariant under (3) and (4). In particular, we conclude that the vector field e_μ determines a $U_I(1)$ principal bundle over $Gr(4,2)$.

We employ $H_{\mu\nu}$ to explicitely resolve for the $U_I(1)$ structure as follows [1]. We first introduce the electric and magnetic components of (11),

$$\begin{aligned}E_i &= \tfrac{i}{2}(e_0 e_i^\star - e_i e_0^\star)\\ B_i &= \tfrac{i}{2}\epsilon_{ijk} e_j^\star e_k\end{aligned} \tag{12}$$

They are subject to

$$\begin{aligned}\vec{E}\cdot\vec{B} &= 0\\ \vec{E}\cdot\vec{E}+\vec{B}\cdot\vec{B} &= \tfrac{1}{4}\end{aligned} \tag{13}$$

We then define the selfdual and anti-self-dual combinations

$$\vec{s}_\pm = 2(\vec{B}\pm\vec{E}) \tag{14}$$

This gives us two independent unit vectors that parametrize the two-spheres $\mathbb{S}^2_\pm$ of our Grassmannian $Gr(4,2)\simeq \mathbb{S}^2_+\times\mathbb{S}^2_-$, respectively. In these variables

$$e_\mu = \frac{1}{2}e^{i\eta}\cdot\left(\sqrt{1-\vec{s}_+\cdot\vec{s}_-}\;,\;\frac{\vec{s}_+\times\vec{s}_- + i(\vec{s}_- - \vec{s}_+)}{\sqrt{1-\vec{s}_+\cdot\vec{s}_-}}\right) \;=\; e^{i\eta}\cdot\left(\sqrt{2\vec{E}\cdot\vec{E}}\;,\;\frac{2\vec{E}\times\vec{B}-i\vec{E}}{\sqrt{2\vec{E}\cdot\vec{E}}}\right) \;\equiv\; e^{i\eta}\hat{e}_\mu \tag{15}$$

Here the phase factor η describes locally a section of the $U_I(1)$ bundle determined by e_μ over the Grassmannian (10). The $U_I(1)$ transformation sends $\eta \to \eta - \lambda$.

We note that since any two components of e_μ can vanish simultaneously, at least three coordinate patches for the base are needed in order to define the bundle. With local trivialization determined by $\eta_\alpha = Arg(e_\alpha)$ these patches can be chosen to be $\mathcal{U}_\alpha = \{|e_\alpha| > \epsilon\}$ for $\alpha = 0, 1, 2$ with some (infinitesimal) $\epsilon > 0$. On the overlaps $\mathcal{U}_\alpha \bigcap \mathcal{U}_\beta$ the transition functions are then

$$f_{\alpha\beta} = \exp\{i \cdot Arg \frac{e_\beta}{e_\alpha}\}$$

with e_α *resp.* e_β a component of vector e_μ that is nonvanishing in the overlap of $\mathcal{U}_\alpha$ and $\mathcal{U}_\beta$.

We now proceed to show by explicit computation the nontriviality of the $U_I(1)$ bundle. This implies that the phase factor η in (15) can not be globally removed. We do this by relating our $U_I(1)$ bundle to the Dirac monopole bundle (Hopf fibration) $\mathbb{S}^3 \sim \mathbb{S}^2 \times \mathbb{S}^1$. We start by introducing the $U_I(1)$ connection

$$\Gamma = i e^\star_\mu d e_\mu \;=\; i \hat{e}^\star_\mu d\hat{e}_\mu + d\eta \;=\; \hat{\Gamma} + d\eta \tag{16}$$

We locally parametrize the vectors $\vec{s}_\pm$ by

$$\vec{s}_\pm \;=\; \begin{pmatrix} \cos\phi_\pm \, \sin\theta_\pm \\ \sin\phi_\pm \, \sin\theta\pm \\ \cos\theta_\pm \end{pmatrix} \tag{17}$$

We substitute this in (15), (16). This gives us a (somewhat complicated) expression of $\hat{\Gamma}$ in terms of the angular variables (17). When we compute the ensuing curvature two-form the result is

$$F = d\hat{\Gamma} \equiv d\Gamma = \sin\theta_+ d\theta_+ \wedge d\phi_+ + \sin\theta_- d\theta_- \wedge d\phi_- \tag{18}$$

Consequently the connection Γ in (16) is gauge equivalent to a connection of the form

$$\Gamma \sim -\cos\theta_+ d\phi_+ - \cos\theta_- d\phi_- + d\eta \tag{19}$$

When we restrict to one of the two-spheres $\mathbb{S}_\pm$ in $Gr(4,2)$ by fixing some point (pt) in the other, we obtain the two submanifolds $\mathbb{S}^2_+ \times pt$ and $pt \times \mathbb{S}^2_-$ and arrive at the functional form of the Dirac monopole connection in each of them. Thus the $U_I(1)$ bundle is non-trivial and admits no global sections, in particular the section η can only be defined locally.

We now proceed to consider the linear action of Euclidean (Lorentz) boosts. For this we rotate e_μ to a generic spatial direction ε_i $(i = 1, 2, 3)$. In the case of an infinitesimal $\varepsilon = \sqrt{\vec{\varepsilon} \cdot \vec{\varepsilon}}$ the four-vector e_μ $(\mu = 0, i)$ transforms under the ensuing boost Λ_ε as follows,

$$\begin{aligned} \Lambda_\varepsilon e_0 &= -\varepsilon_i e_i \\ \Lambda_\varepsilon e_i &= -\varepsilon_i e_0. \end{aligned} \tag{20}$$

For a finite ε the boost is obtained by exponentiation,

$$\begin{aligned} e^{\Lambda_\varepsilon}(e_i) \;&=\; e_i + \tfrac{1}{\varepsilon^2} \cdot \varepsilon_i (\vec{e} \cdot \vec{\varepsilon} \cos(\varepsilon) + \varepsilon e_0 \sin(\varepsilon) - \vec{e} \cdot \vec{\varepsilon}) \\ e^{\Lambda_\varepsilon}(e_0) \;&=\; e_0 \cos(\varepsilon) - \tfrac{1}{\varepsilon} \vec{e} \cdot \vec{\varepsilon} \sin(\varepsilon) \;\equiv\; e_\mu \hat{\varepsilon}_\mu \end{aligned} \tag{21}$$

where

$$0 \leq \; \varepsilon \equiv \sqrt{\vec{\varepsilon} \cdot \vec{\varepsilon}} \; < 2\pi \qquad (mod \; 2\pi)$$

and

$$\hat{\varepsilon}_\mu = \left(\cos(\varepsilon) \;, \; -\sin(\varepsilon) \frac{\vec{\varepsilon}}{\varepsilon} \right)$$

We now identify a different, *projective* representation of $SO(4)$ on the Grassmannian: On the base manifold the ensuing $SO(4)$ boost acts on the electric and magnetic vectors $\vec{E}$ and $\vec{B}$ so that the result is the familiar

$$\begin{aligned} \Lambda_\varepsilon \vec{E} \equiv \delta_\varepsilon \vec{E} = \vec{B} \times \vec{\varepsilon} \\ \Lambda_\varepsilon \vec{B} \equiv \delta_\varepsilon \vec{B} = \vec{E} \times \vec{\varepsilon}. \end{aligned} \tag{22}$$

For finite boost we get

$$e^{\delta_\varepsilon}(\vec{E}) = \frac{\vec{\varepsilon}(\vec{\varepsilon}\cdot\vec{E})\,(1-\cos\varepsilon) + [\vec{B}\times\vec{\varepsilon}]\,\varepsilon\sin\varepsilon + \vec{E}\,\varepsilon^2\cos\varepsilon}{\varepsilon^2} \tag{23}$$

and the same holds for the finite boost of $\vec{B}$, but with $\vec{E}$ and $\vec{B}$ interchanged.

We assert that the difference between (20) and (22), *resp.* (21) and (23), is a one-cocycle, due to the projective nature of the second representation of $SO(4)$ on $Gr(4,2)$. For this we recall the definition of a one-cocycle: If ξ denotes a local coordinate system on $Gr(4,2)$ and if a section of the $U_I(1)$ bundle which is locally specified by $e^{i\eta}$ is denoted by Ψ, then we have for a projective representation

$$\Lambda(g)\Psi(\xi) = \mathcal{C}(\xi,g)\Psi(\xi^g) \tag{24}$$

with $g \in SO(4)$. The factor $\mathcal{C}(\xi,g)$ is a one-cocycle that determines the lifting of the projective representation to the linear representation. For a boost with the group element $g \in SO(4)$ which is parameterized by (finite) $\vec{\varepsilon}$ on the base manifold with $\vec{E}$ and $\vec{B}$, (24) becomes

$$e^{\Lambda_\varepsilon}\Psi(\vec{E},\vec{B}) = \mathcal{C}(\vec{E},\vec{B},\vec{\varepsilon})\Psi\left(e^{\delta_\varepsilon}(\vec{E}), e^{\delta_\varepsilon}(\vec{B})\right) \tag{25}$$

We compute the one-cocycle in (25) on a chart $\mathcal{U}_0$ with local trivialization $\eta = Arg(e_0)$. With $C(\xi,g) = \exp\{i\Theta(\xi,g)\}$ we look at the transformation of a local section $\exp\{\eta\}$ under the boost g. Under an infinitesimal boost the phase of e_0 changes as follows [1],

$$\Lambda_\varepsilon\eta = \Theta(\varepsilon) = \frac{\vec{E}\cdot\vec{\varepsilon}}{2\vec{E}^2} = \frac{(\vec{s}_+ - \vec{s}_-)\cdot\vec{\varepsilon}}{1-\vec{s}_+\cdot\vec{s}_-} \tag{26}$$

For a finite boost we find by exponentiation

$$\Theta(\vec{\varepsilon}) \;=\; Arg\,(\hat{e}_\mu\hat{\varepsilon}_\mu) \tag{27}$$

which reduces to (26) for infinitesimal ϵ. For general $g \in SO(4)$ we get in the chart $\mathcal{U}_0$

$$\Theta(\xi,g) = Arg\left(\frac{e_0^g}{e_0}\right) \tag{28}$$

Finally, since all one-dimensional representations are necessarily Abelian we conclude that Θ satisfies the one-cocycle condition

$$\Lambda_{\varepsilon_1}\Theta(\vec{E},\vec{B};\vec{\varepsilon}_2) \;-\; \Lambda_{\varepsilon_2}\Theta(\vec{E},\vec{B};\vec{\varepsilon}_1) \;=\; 0$$

with its nontriviality following from the nontriviality of the Dirac monopole bundles.

In conclusion, we have established the nontriviality of the infinitesimal one-cocycle found in [1] by relating it to the Dirac monopole bundle. We have also reported its finite version. The presence of the one-cocycle establishes that in spin-charge separated Yang-Mills theory Lorentz boosts have two inequivalent representations, one acting linearly on the Grassmannian $Gr(2,4)$ and the other projectively. The physical consequences of this observation remain to be clarified; in [1] a relation to Yang-Mills mass gap has been proposed.

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